Due Sun

5.4 – Differential Equations

2nd-order DE

$y'' - 6y' - 5y = 0$

Definitions: A **differential equation** is an equation involving unknown functions and their derivatives.

The **order** of a differential equation is the order of the highest derivative it contains.

$10 \cdot 3e^{3x} = 3(10e^{3x})$

$y = e^{3x}$, $y = 10e^{3x}$

1st-order DE.

A differential equation of the form $y' = ay$ has a **general solution** of the form $y = ce^{ax}$.

If $\frac{dy}{dx} = ay$, $\int \frac{dy}{y} = \int a\,dx \Rightarrow \ln|y| = ax + C_1$

$|y| = e^{ax+C_1} \Rightarrow |y| = e^{C_1}e^{ax}$

$y = ce^{ax}$

A condition which specifies the value of the general solution at a point is called an **initial condition**, and the problem of solving a differential equation subject to an initial condition is called an **initial-value problem**.

If we know, for instance, that $y(0) = P_0$, then $y = P_0 e^{ax}$.

A **constant coefficient first-order homogeneous linear system** has the form

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\
\vdots & \qquad \vdots \qquad \vdots \qquad\quad \vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n
\end{aligned}$$

where $y_i = f_i(x)$ are functions to be determined, and the a_{ij}'s are constants.

This can be written in matrix notation as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or $\mathbf{y}' = A\mathbf{y}$.

The **trivial solution** to the above differential equation is $y_1 = y_2 = \ldots = y_n = 0$.

If we can find a matrix P that diagonalizes A, then we can use the diagonal matrix in solving the system. If P diagonalizes A, then we form $\vec{y} = P\vec{u}$, where $\vec{u}$ is an unknown vector of functions.

Then $\vec{y}\,' = A\vec{y} \Rightarrow P\vec{u}\,' = AP\vec{u}$ diagonal

$$\Rightarrow \vec{u}\,' = P^{-1}AP\,\vec{u} = D\vec{u}.$$

This has the form of a system that can be solved using exponentials.

Recap: Imagine an unknown vector $\vec{u}$ of functions exists. Then find $\vec{u}$ and use $\vec{y} = P\vec{u}$.

$$P = [\vec{P}_1\ \vec{P}_2 \cdots \vec{P}_n],\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

Example:

a. Solve the system of differential equations.

$y_1' = 2y_1 + 3y_2$

$y_2' = 2y_1 + y_2$

- Find eigenvalues and eigenvectors to form P and D
- Solve $\vec{u}' = D\vec{u}$
- Find $\vec{y} = P\vec{u}$

5.1

$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \Rightarrow |\lambda I - A| = 0$ gives us

$\begin{vmatrix} \lambda-2 & -3 \\ -2 & \lambda-1 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 - 6 = 0$

$\lambda = 4, -1$

$\lambda = 4: \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \rightarrow \vec{p}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda = -1: \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \rightarrow \vec{p}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

5.2

$P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

5.4

$\vec{u}' = D\vec{u} \Rightarrow \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} 4u_1 \\ -u_2 \end{bmatrix}$

$u_1' = 4u_1 \rightarrow u_1(x) = c_1 e^{4x}$

$u_2' = -u_2 \quad\; u_2(x) = c_2 e^{-x}$

$\vec{y} = P\vec{u} \Rightarrow$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{4x} \\ c_2 e^{-x} \end{bmatrix} = \begin{bmatrix} 3c_1 e^{4x} + c_2 e^{-x} \\ 2c_1 e^{4x} - c_2 e^{-x} \end{bmatrix}$

$$\boxed{\begin{aligned} y_1(x) &= 3c_1 e^{4x} + c_2 e^{-x} \\ y_2(x) &= 2c_1 e^{4x} - c_2 e^{-x} \end{aligned}}$$

b. Find the solution that satisfies the initial conditions $y_1(0) = 25$, $y_2(0) = 5$.

$y_1(0) = 25 \Rightarrow \quad 25 = 3c_1 + c_2 \qquad c_1 = 6$

$y_2(0) = 5 \Rightarrow \quad 5 = 2c_1 - c_2 \qquad c_2 = 7$

$$\boxed{\begin{aligned} y_1 &= 18e^{4x} + 7e^{-x} \\ y_2 &= 12e^{4x} - 7e^{-x} \end{aligned}}$$

#8 Use the procedure in Exercise 7 to solve $y'' + y' - 12y = 0$.

$y_1' = ay_1 + by_2$
$y_2' = cy_1 + dy_2$
• One derivative per equation
• 1st-order DE

Let $y_1 = y$ and $y_2 = y' = y_1'$

Then $y_2' + y_2 - 12y_1 = 0$

$y_1' = y_2$
$y_2' = 12y_1 - y_2$
$\rightarrow \quad \vec{y}' = \begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix} \vec{y}$

$\begin{vmatrix} \lambda & -1 \\ -12 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda - 12 = 0$

$\lambda_1 = -4: \begin{bmatrix} -4 & -1 \\ -12 & -3 \end{bmatrix} \rightarrow \vec{P}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\lambda_2 = 3: \begin{bmatrix} 3 & -1 \\ -12 & 4 \end{bmatrix} \rightarrow \vec{P}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$

$\vec{u}' = D\vec{u} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$u_1' = -4u_1 \Rightarrow u_1 = c_1 e^{-4x}$

$u_2' = 3u_2 \Rightarrow u_2 = c_2 e^{3x}$

$$\vec{y} = P\vec{u} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} c_1 e^{-4x} \\ c_2 e^{3x} \end{bmatrix}$$

$$y = \cancel{y_1} = c_1 e^{-4x} + c_2 e^{3x}$$

y_2 = we don't care

Check: $y_2 = -4c_1 e^{-4x} + 3c_2 e^{3x}$

match

AND $y_2 = y_1' = -4c_1 e^{-4x} + 3c_2 e^{3x}$